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Crystallography, Geometry and Physics in Higher Dimensions. XI. A New Geometrical Method for Systematic Construction of the *n*-Dimensional Crystal Families: Reducible and Irreducible Crystal Families

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Abstract

A geometrical method is described for counting and constructing all crystal families of Euclidean spaces of dimension n from the different well known crystal families of spaces E^1, E^2, E^3 and the geometrically Z-irreducible families of each space. The definition of the geometrically Z-irreducible (gZ-irr.) or geometrically Z-reducible (gZ-red.) families is connected to the properties of the character table of the holohedry of these families. Indeed, a crystal family of space E^n is said to be gZ-irr. if the n translation operators corresponding to a basis of a primitive Bravais cell belong to the same irreducible representation with integer entries of its holohedry. In the opposite case, the family is said to be gZ-red. This method enables a name to be given to the crystal families. This name is connected to the geometrical construction, except for the families considered as irreducible. As far as possible, it also recalls the name of the crystal families of spaces E^1, E^2 and E^3. Moreover, the WPV (Weigel–Phan–Veysseyre) symbols of the holohedries can be defined thanks to the properties of the crystal cells. Finally, all the point-symmetry operations of these groups and subgroups can be listed.

Introduction

In the Euclidean space E^1 (of dimension 1), there is only one crystal family: its cell is a (straight-line) segment, the symbol of the holohedry is m and the two point-symmetry operations (PSOs) are the identity (1) and the symmetry about a point mirror (m). This operation maps the point x onto $\bar{x}$, so it could be called $\bar{1}$. In fact, in one-dimensional space, the mapping 'symmetry about a point', *i.e.* $\bar{1}$, is the same as the reflection through a point mirror (m). In two-dimensional space, the mirror is a straight line and in three-dimensional space, it is a plane.

Now let us consider the Euclidean space E^2 (of dimension 2). There are four crystal families: the rectangle family, in which the cell is a rectangle; the oblic family, in which the cell is a parallelogram; the square (or tetragon) family, in which the cell is a square; and the hexagon family,* in which the cell is a hexagon.

The 'rectangle' cell can be considered as the rectangular product of two cell 'segments' belonging to two orthogonal subspaces of dimension 1.

As a consequence, we assume that the rectangle family is reducible of type 1 + 1 because $E^2 = E^1 \oplus E^1$ (see § I). We recommend $m \perp m$ for the symbol of the holohedry instead of $2mm$. With respect to the basis defined by the two sides of the rectangle cell, the metric tensor† is

$$\left(\begin{array}{c|c} a & 0 \\ \hline 0 & b \end{array}\right),$$

in which a and b are the squared norms of the two sides, $\mathbf{a}_1$ and $\mathbf{a}_2$, of the cell, *i.e.* $a = \|\mathbf{a}_1\|^2$, $b = \|\mathbf{a}_2\|^2$. The splitting up of the tensor is obvious.

The symbol $\perp$ in $m \perp m$ means 'orthogonal' if we consider the geometric properties of the cell and 'direct product' if we consider the algebraic properties of the group; this symbol immediately gives the order of the point-symmetry group (PSG), *i.e.* $2 \times 2 = 4$ as well as all the PSOs, namely m_x, m_y, 2_{xy}, 1, where x, y are the axes defining the vectors of the cell $\mathbf{a}_1$, $\mathbf{a}_2$.

The other three families are described as 'geometrically Z-irreducible' (abbreviated to gZ-irr.). This property will be explained in § II.

Let us now consider the Euclidean space E^3 (of dimension 3). The six crystal families can be constructed in a similar way.

(i) One cell is the rectangular product of cells belonging to three subspaces that are orthogonal two-by-two, *i.e.* three copies of space E^1. This family is

* The same name must not be used for two different crystal families; hexagonal must be kept only for a crystal family of E^3.

† Definition of a metric tensor. Let $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ be the n vectors describing a primitive Bravais cell of a crystal family of space E^n. Then the metric tensor has for entries the scalar products $\mathbf{a}_i \cdot \mathbf{a}_i, \ldots$, *i.e.* the squared norms of $\mathbf{a}_i$ for the main diagonal: $\mathbf{a} = \|\mathbf{a}_1\|^2$, $\mathbf{b} = \|\mathbf{a}_2\|^2, \ldots$ and the scalar product of two different vectors for the other entries:

$$A = \mathbf{a}_1 \cdot \mathbf{a}_2 = \|\mathbf{a}_1\| \, \|\mathbf{a}_2\| \cos(\mathbf{a}_1, \mathbf{a}_2), \ldots,$$

where $\| \ \|$ means the norm.

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called 'orthorhombic' in classical crystallography and $2/mmm$ is the symbol of its holohedry. Nevertheless, to generalize to higher-dimensional spaces, the best name would be 'orthotopic $3d$' (orthotope is the name of the generalized rectangle) and $m \perp m \perp m$ would be the best symbol for the holohedry of this family. As previously, this symbol gives the order of the PSG, $2\times 2\times 2=8$; it enables us easily to define its elements: m_x; m_y; m_z; 2_{xy}; 2_{xz}; 2_{yz}; $\bar{1}_{xyz}$; 1; *i.e.* three reflections in a mirror plane, three twofold rotations in the planes xy, xz and yz, one homothetie of ratio -1 and of dimension 3 and the identity. This enables us to define the subgroups of this group as follows: three subgroups of order 4: $m \perp m$, 2,2,2, $2 \perp m$; three subgroups of order 2: m, 2, $\bar{1}$; one subgroup of order 1: 1. The metric tensor is

$$\left(\begin{array}{c|c|c} a & 0 & 0 \\ \hline 0 & b & 0 \\ \hline 0 & 0 & c \end{array}\right),$$

where a, b and c are the squared norms of the sides of the cell. Owing to these properties, this family is reducible of type $1+1+1$ ($E^3=E^1\oplus E^1\oplus E^1$).

(ii) Three crystal cells can be regarded as the rectangular product of the crystal cell of the space E^1 (segment) and one of the three gZ-irr. cells of space E^2. The crystal cells built in this way are 'right prisms based on geometrically Z-irreducible crystal cells' of space E^2. This very long name, which fully explains the structure of the cell, is shortened to '-al', which is the abbreviation for 'orthogonal'. Consequently, the names of the families are: 'oblical': crystal family whose cell is a right prism based on a parallelogram; tetragonal: crystal family whose cell is a right prism based on a square; hexagonal: crystal family whose cell is a right prism based on a hexagon. We suggest the following WPV symbols* for the respective holohedries: $2\perp m$ instead of $2/m$, order $2\times 2=4$; $4mm\perp m$ instead of $4/mmm$, order $4\times 2=8$; $6mm\perp m$ instead of $6/mmm$, order $6\times 2=12$. The classical names of these families are: monoclinic; tetragonal; hexagonal. These crystal families are geometrically Z-reducible of type $2+1$ ($E^3=E^2\oplus E^1$).

(iii) Finally, we add two gZ-irr. families, triclinic and cubic, obviously of type 3.

Before proceeding, two approaches will be discussed:

(1) consideration of every possible partition of space E^n into subspaces E^p, two-by-two orthogonal, of dimension p less than n;

(2) definition of every gZ-irr. family of space E^p for all values of the integer p less than or equal to n.

* Weigel–Phan–Veysseyre symbols (Weigel, Phan & Veysseyre, 1986).

I. Geometric method for constructing crystal families

Point (1) can be dealt with easily. All the partitions of integer n (dimension of space) are written as sums of positive integers. For instance, for number 6, there are 11 partitions: 6; 5+1; 4+2; 4+1+1; 3+3; 3+2+1; 3+1+1+1; 2+2+2; 2+2+1+1; 2+1+1+1+1; 1+1+1+1+1+1. As an example, let us consider the partition $6=3+2+1$, *i.e.* the splitting up of space E^6 into $E^3\oplus E^2\oplus E^1$. Space E^3 contains two geometrically Z-irreducible crystal cells: triclinic and cubic. In space E^2, there are three of them: oblic, square and hexagon. Therefore, in the five-dimensional space, the partition $E^3\oplus E^2$ leads to $2\times 3=6$ crystal cells. Owing to their construction, these are called triclinic oblic, triclinic square, triclinic hexagon, cubic oblic, cubic square and cubic hexagon. In fact, the first name is the abridged form of orthogonal triclinic oblic, which means that the two polytopes are subcells belonging to two orthogonal subspaces.

Now, we must add space E^1 (to give $E^3\oplus E^2\oplus E^1$). This results in six crystal families whose cells are right hyperprisms based on the above-mentioned six crystal cells. For these six crystal families, we suggest a shortened name: triclinic oblic-al; cubic square-al; and so on. Hence, we can define the WPV symbols of the holohedries of all these families. As an example, the holohedry of the crystal family 'triclinic square-al' of the six-dimensional space is

$\bar{1}\perp 4mm\perp m$ of order $2\times 8\times 2=32$

triclinic square right hyperprism based on ....

If (x, y, z, t, u, v) denotes the orthogonal basis of space E^6 connected to the different partial cells, the PSOs of this group are: 1, $\bar{1}_{xyz}$ for the PSG $\bar{1}$; 1, $4_{tu}^{\pm 1}$, 2_{tu}, m_{t+u}, m_{t-u}, m_t, m_u for the PSG $4mm$; 1, m_v for the PSG m; and all the products of these elements.

Now, let us consider the partition $E^3\oplus E^3$. The number of crystal cells corresponding to this partition is the number of combinations with repetitions of two elements (the 2 gZ-irr. crystal cells of space E^3) taken two at a time, *i.e.*

$$\binom{2+2-1}{2}=\binom{3}{2}=3.$$

These are triclinic-triclinic or di triclinic, triclinic-cubic and di cubic. The WPV symbols of the holohedries are, respectively:

(i) $\bar{1}\perp\bar{1}$, of order $2\times 2=4$ (PSOs: $\bar{1}_{xyz}$, $\bar{1}_{tuv}$, $\bar{1}_6$, 1);

(ii) $\bar{1}\perp m\bar{3}m$, of order $2\times 48=96$;

(iii) $m\bar{3}m\perp m\bar{3}m$, of order $48^2=2304$.

All partitions of space E^6 are to be studied similarly.

To finish with, we can give an example of a crystal family of space E^7 corresponding to the decomposition $E^3\oplus E^4$. In space E^3, we select the cubic family and in space E^4, the rhombotopic $(-\frac{1}{4})$ family. The cell of the rhombotopic family is constructed from four equal vectors and the angle between two of them has a cosine of $-\frac{1}{4}$ (Phan, Veysseyre & Weigel, 1988). The orthogonal product of these two polytopes gives a cell named 'cubic rhombotopic $(-\frac{1}{4})$'. The WPV symbol of the holohedry is $m\bar{3}m \perp \bar{4}3m$, 10_2, of order $48 \times 240 = 11\,520$. We could describe these 11 520 PSOs without difficulty but this would take a long time!

II. Geometrical Z-irreducible families

A mathematical definition of the K-irreducibility or the K-reducibility (K being one of the domains Z, Q, R or C) is given by Brown, Bülow, Neubüser, Wondratschek & Zassenhaus (1978). It is connected to the unimodular $n \times n$ matrices of finite groups associated to a Z class.

We use a slightly different concept of irreducibility connected to the geometry and the splitting up of the metric tensor of a crystal cell. For this reason, the adjective 'geometrical' and the letter Z (Z being the set of positive or negative integers) are added to the word 'irreducibility'. Therefore, we define the geometrical Z-irreducibility of the holohedry of a crystal family (*i.e.* of the point group of maximal order). Actually, we define the geometrical Z-irreducibility of the cell of this family (gZ-irr. for short) and we will say that a family (or its cell) is gZ-irr. or gZ-red. in the opposite case. This definition is connected to the bases of the irreducible representations of the holohedry of the family (Weigel & Veysseyre, 1991).

Let $x, y, z, t, u, v, \ldots$ be the n translation operators corresponding to a basis of a primitive Bravais cell of a crystal family of the n-dimensional space E^n. This family is said to be geometrically Z-irreducible (gZ-irr.) if all these operators belong to the same irreducible representation (IR) with integer entries of its holohedry. In the opposite case, the family is said to be geometrically Z-reducible (gZ-red.) Let us start by describing three simple cases of the space E^2.

(i) The holohedry of the rectangular family is the group $m \perp m$ of order 4. Let x, y be the basis of a primitive cell. The character table of group $m \perp m$ is given in Table 1. There are four IR of dimension 1. Thanks to the theory of projectors, it is easy to prove that x belongs to the IR R_2 and y to the IR R_3. We say that this family is gZ-red. of type $1+1$; it agrees with the correct symbol $m \perp m$ of the holohedry.

(ii) The holohedry of the oblic family is the group 2 of order 2. The character table of this group is given in Table 2. In this case, the two translation operators x and y belong to the same IR R_2, which is not the identity representation. We say that this family is gZ-irr. of type $\overline{1,1}$.

(iii) Now, we consider the square family; its holohedry is $4mm$ of order 8 and the character table

Table 1. *Character table of point group $m \perp m$*

First line: the four classes of conjugate element of this group.
E: identity.
m_x (and m_y): reflection through a straight line.
2_{xy}: twofold rotation in the plane (xy).

$m \perp m$	E	m_x	m_y	2_{xy}	
R_1	1	1	1	1	
R_2	1	−1	1	−1	x
R_3	1	1	−1	−1	y
R_4	1	−1	−1	1	

Table 2. *Character table of point group* 2

See caption of Table 1.

2	E	2_{xy}	
R_1	1	1	
R_2	1	−1	x, y

Table 3. *Character table of point group $4mm$*

First line: the five classes of conjugate elements of this group.
$2c_4 = \{4^1_{xy}, 4^{-1}_{xy}\}$ (fourfold rotation in the plane xy),
$2\sigma = \{m_x, m_y\}$, $2\sigma' = \{m_{x+y}, m_{x-y}\}$.

$4mm$	E	$2c_4$	C_2	2σ	$2\sigma'$	
A_1	1	1	1	1	1	
A_2	1	1	1	−1	−1	
B_1	1	−1	1	1	−1	
B_2	1	−1	1	−1	1	
E	2	0	−2	0	0	(x, y)

is given in Table 3. The two translation operators x and y belong to the same IR of dimension 2. We say that this family is gZ-irr. of type 2.

In space E^2, it is obviously the only possible way of splitting up. The four crystal families are classified as follows: the rectangle family is gZ-red. of type $1+1$; the oblic family is gZ-irr. of type $\overline{1,1}$; the square family and the hexagon family are gZ-irr. of type 2.

However, if the dimension of space increases, the number of different types of gZ-irr. increases too and, in a space of even dimension, this number quickly increases owing to the occurrence of new point-symmetry operations such as double or triple rotations $[5]_2, [7]_3, \ldots$ (Weigel, Veysseyre, Phan, Effantin & Billiet, 1984) and also to the possibilities for two planes or two subspaces to be monoclinic or diclinic. In contrast, the case of spaces of odd dimension is easier. Indeed, from space E^4 to space E^5, for instance, we add only one dimension; then the only possibilities for new crystal cells are 'right hyperprisms based on . . .'.

Let us study space E^4 in detail, so that we may list the five types of gZ-irreducibility. As previously, (x, y, z, t) denotes the basis connected to a crystal cell of space E^4.

(i) We first consider the character table of the hexaclinic family whose WPV symbol for the holohedry is $\bar{1}_4$, of order 2; the four vectors (x, y, z, t)

Table 4. *Character table of point group* 2, 66*, 2 *of order* 12

First line: the six classes of conjugate elements of this group.

$C_1=\{E\}=$ identity,
$C_2=\{\bar{1}_3\}$,
$3C_3=\{2_{x,z-2t};\ 2_{y,2z-t};\ 2_{x-y};\ z+t\}$,
$3C_4=\{2_{x-2y};\ z;\ 2_{2x-y};\ t;\ 2_{x+y};\ z+t\}$,
$2C_5=\{3^1_{xy}3^1_{\gamma\delta};\ 3^{-1}_{xy}3^{-1}_{\gamma\delta}\}$,
$2C_6=\{6^1_{xy}6^1_{\gamma\gamma};\ 6^{-1}_{xy}6^{-1}_{\gamma\delta}\}$.

For a complete explanation of these symbols, see Phan *et al.* (1988). $6^1_{xy}6^1_{\gamma\delta}$ is a double crystallographic rotation: by $2\pi/6$ in the plane xy and $2\pi/6$ in the plane $\gamma\delta$ (supplementary and orthogonal to the plane xy).

2, 66*, 2	C_1	C_2	$3C_3$	$3C_4$	$2C_5$	$2C_6$	
R_1	1	1	1	1	1	1	
R_2	1	1	−1	−1	1	1	
R_3	1	−1	1	−1	1	−1	
R_4	1	−1	−1	1	1	−1	
R_5	2	−2	0	0	−1	1	$(x, y)(x, t)$
R_6	2	2	0	0	−1	−1	

belong to the same IR of dimension 1, which is not the identity representation. We say that this family is gZ-irr. of type $\overline{1,1,1,1}$. In space E^n, whatever the dimension n, only one crystal family has this property: it is the 'maxiclinic' family whose name recalls the number of angular parameters of the metric tensor of the cell: oblic (E^2), triclinic (E^3), hexaclinic (E^4), decaclinic (E^5),.... This type is denoted $\overline{1,1,\ldots,1}$ with n numericals 1 for space E^n.

(ii) We next consider the character table of the holohedry of the hypercubic family. The four vectors (x, y, z, t) belong to an IR of dimension 4: the IR labelled R_{11} (Veysseyre, Weigel, Phan & Effantin, 1984). We say that this crystal family is gZ-irr. of type 4. Another two crystal families have this property: the rhombotopic $(-\frac{1}{4})$ and the di isohexagon families (Phan *et al.*, 1988). In space E^n, this type will be denoted n and it means that the n vectors $(x, y, z, \ldots)$ belong to the same n-dimensional IR with integer entries. In the space E^5 there are two families of this type: rhombotopic $(-\frac{1}{5})$ and hypercubic five-dimensional, whereas there are three in space E^6: tri iso hexagon, rhombotopic $(-\frac{1}{6})$ and hypercubic six-dimensional.

(iii) Let us now study the character table of the monoclinic di hexagon family (Table 4). The WPV symbol of the holohedry is 2, 66*, 2. The vectors (x, y) or the vectors (z, t) are possible bases of the IR of dimension 2 labelled R_5. We say that this family is gZ-irr. of type $\overline{2,2}$. Another crystal family belongs to this type: the monoclinic di square family. In space E^6, three families belong to this type, denoted $\overline{2,2,2}$ (monoclinic tri square and monoclinic tri hexagon families) or $\overline{3,3}$ (monoclinic di cubic family).

(iv) The character table of the diclinic di hexagon family of E^4 is given in Table 5. In this case, the same combinations $x-\frac{1}{2}(1+i3^{1/2})y$; $z-\frac{1}{2}(1+i3^{1/2})t$ of the four vectors (x, y, z, t) belong to the same IR with complex entries, whereas the combinations $x-\frac{1}{2}(1-i3^{1/2})y$ and $z-\frac{1}{2}(1-i3^{1/2})t$ belong to an IR that is the complex conjugate of the previous one. Then, (x, y) and (z, t) are possible bases of a gZ-irreducible representation with integer entries $R_5\oplus R_6$, which is the direct sum of the two previous ones. We suggest for this type of gZ-irr. the notation $\overline{2,2}'$. Another family of space E^4 belongs to this type (the diclinic di square family) and two families of space E^6 (diclinic tri square, diclinic tri hexagon families).

(v) The last type of irreducibility is exemplified by the monoclinic di isohexagon family, of space E^4. The WPV symbol of its holohedry is 12 12 ∧ 2 and its order is 24 (see its character table, Table 6). Thanks to the theory of projectors, we find linear combinations of the four vectors (x, y, z, t) with irrational entries as possible bases for two IR, namely R_8 and R_9, but (x, y, z, t) are the bases of the direct sum of these two, *viz* $R_8\oplus R_9$. For this reason, we say that this family is gZ-irr. of type 4′.

As a conclusion, we summarize the different types of irreducibility. We recall that $(x, y, z, \ldots)$ are the bases of the crystal cell. The character table is that of the holohedry.

Type 1. Denoted $\overline{1,1,1,\ldots,1}$ (n numbers 1 in space E^n): the n vectors $x, y, z, \ldots$ belong to the same one-dimensional IR (not the identity IR).

Type 2. Denoted n in space E^n: the n vectors belong to one n-dimensional IR with integer entries.

Type 3. Denoted $\overline{2,2}$ in space E^4, $\overline{2,2,2}$ or $\overline{3,3}$ in space E^6 and so on. In space E^4, for instance, the vectors (x, y) and (z, t) are possible bases of the same two-dimensional IR with integer entries.

Type 4. Denoted $\overline{2,2}'$ in space E^4 or $\overline{2,2,2}'$ in space E^6.... In space E^4, for instance, the linear combinations with complex entries of vectors (x, y) and (z, t) belong to two complex conjugate IRs of dimension 1. Therefore, (x, y) and (z, t) are possible bases of a two-dimensional IR with integer entries, the direct sum of the two previous ones.

Type 5. Denoted n' in space E^n. The definition is similar to the previous one but we consider linear combinations of vectors (x, y) (z, t)... with irrational entries and we consider IRs with irrational entries whose direct sum is an n-dimensional IR with integer entries.

All the irreducible families (and the reducible ones) will be listed in tables in papers XII, XIII and XIV for each type of irreducibility and for each space $E^1 \ldots E^7$.

III. Application: how to define the type of a gZ-red. family

The classification of the gZ-irr. families enables us to study the gZ-red. families and to define their type. We explain this method through the example of the

Table 5. *Character table of the cyclic point group* 66* *of order* 6

First line: the six classes of conjugate elements.
$e = \exp(2i\pi/6) = \frac{1}{2}(1 + i\sqrt{3})$
$\bar{1}_4$ = homothetie of ratio -1 and of dimension 4.

66*	E	$\bar{1}_4$	$3^{-1}3^{-1}$	$3^1 3^1$	$6^{-1}6^{-1}$	$6^1 6^1$	
R_1	1	1	1	1	1	1	
R_2	1	-1	1	1	-1	-1	
R_3	1	1	e^2	e^4	e^4	e^2	
R_4	1	1	e^4	e^2	e^2	e^4	
R_5	1	-1	e^2	e^4	e	e^5	$x - \frac{1}{2}(1 + i3^{1/2})y$; $z - \frac{1}{2}(1 + i3^{1/2})t$
R_6	1	-1	e^4	e^2	e^5	e	$x - \frac{1}{2}(1 - i3^{1/2})y$; $z - \frac{1}{2}(1 - i3^{1/2})t$
$R_5 \oplus R_6$	2	-2	-1	-1	1	1	$(x, y)(z, t)$

Table 6. *Character table of point group* $12\,12 \wedge 2$

First line: the nine classes of conjugate elements of this group.
$6C_2 = \{6$ twofold rotations$\}$. $2C_6 = \{4^1 4^1; 4^{-1}4^{-1}\}$,
$6C_3 = \{6$ twofold rotations$\}$, $2C_7 = \{6^1 6^1; 6^{-1}6^{-1}\}$,
$2C_5 = \{4^1 4^1; 4^{-1}4^{-1}\}$, $2C_8 = \{3^1 3^1; 3^{-1}3^{-1}\}$.

$12\,12 \wedge 2$	E	$6C_2$	$6C_3$	$\bar{1}_4$	$2C_5$	$2C_6$	$2C_7$	$2C_8$	$2C_9$	
R_1	1	1	1	1	1	1	1	1	1	
R_2	1	1	-1	1	-1	1	1	-1	-1	
R_3	1	-1	-1	1	1	1	1	1	1	
R_4	1	-1	1	1	-1	1	1	-1	-1	
R_5	2	0	0	2	-2	-1	-1	1	1	
R_6	2	0	0	2	2	-1	-1	-1	-1	
R_7	2	0	0	-2	0	2	-2	0	0	
R_8	2	0	0	-2	0	-1	1	$3^{1/2}$	$-3^{1/2}$	
R_9	2	0	0	-2	0	-1	1	$-3^{1/2}$	$3^{1/2}$	
$R_8 \oplus R_9$	4	0	0	-4	0	-2	2	0	0	(x, y, z, t)

Table 7. *Character table of point group* $2 \perp \bar{1} \perp m$

First line: classes of conjugate elements.
2_{xy} is the rotation through angle π in the plane xy.
$\bar{1}_{ztu}$ is the three-dimensional inversion in the space (ztu).
$\bar{1}_{xyztu}$ is the five-dimensional inversion in the space $xyztu$.
$\bar{1}_6$ is the total homothetie of ratio (-1) in the six-dimensional space.

$2 \perp \bar{1} \perp m$	E	2_{xy}	$\bar{1}_{ztu}$	$\bar{1}_{zyztu}$	m_v	$\bar{1}_{xyv}$	$\bar{1}_{ztuv}$	$\bar{1}_6$	
R_1	1	1	1	1	1	1	1	1	
R_2	1	-1	1	-1	1	-1	1	-1	x, y
R_3	1	1	-1	-1	1	1	-1	-1	z, t, u
R_4	1	-1	-1	1	1	-1	-1	1	
R_5	1	1	1	1	-1	-1	-1	-1	v
R_6	1	-1	1	-1	-1	1	-1	1	
R_7	1	1	-1	-1	-1	-1	1	1	
R_8	1	-1	-1	1	-1	1	1	-1	

oblic triclinic family of space E^6. The WPV symbol of its holohedry is $2 \perp \bar{1} \perp m$ and its order is $2 \times 2 \times 2 = 8$. Indeed, 2 is the symbol of the holohedry of the oblic family in E^2, $\bar{1}$ is the symbol of the holohedry of the triclinic family in E^3, hence $\bar{1} \perp m$ is the symbol of the holohedry of the triclinic-al family in E^4 whose cell is the right hyperprism based on the oblic parallelepiped of the physical space. The eight PSOs are: 1; 2_{xy}; $\bar{1}_{ztu}$; m_v; $\bar{1}_{xyztu}$; $\bar{1}_{xyv}$; $\bar{1}_{ztuv}$; $\bar{1}_6$. The three subgroups 2, $\bar{1}_3$ and m are isomorphic abstract groups. Therefore, they have the same abstract character table, Table 2. The character table of group $2 \perp \bar{1} \perp m$ is easily obtained as the tensorial product of three similar character tables of Table 2; it is given in Table 7.

x or y are possible bases of the one-dimensional IR R_2, hence the oblic family is gZ-irr. of type $\overline{1,1}$. z, t or u are a possible basis of the one-dimensional IR R_3, hence the triclinic family is gZ-irr. of type $\overline{1,1,1}$; v is the basis of the one-dimensional IR R_5, hence this family is gZ-irr. of type 1. Therefore, the oblic triclinic-al family is gZ-red. of type $\overline{1,1} + \overline{1,1,1} + 1$. It corresponds to the splitting up of space E^6 into $E^6 = E^2 \oplus E^3 \oplus E^1$.

All crystal families of space E^n, whatever the dimension n, can be studied similarly.

Concluding remarks

In § I, we explained a geometric method for constructing crystal families of space E^n out of the different families of spaces E^1 and E^2 and the gZ-irr. families. The irreducible families were studied in detail in § II and five types of irreducibility were exhibited.

This method enables us to give a name to the crystal family connected to their construction. Then, we easily deduce the WPV symbols of the holohedries. As for the gZ-irr. family, the name explains their construction or sometimes the PSOs that characterize the families.

In forthcoming papers, we state systematic rules for giving a name to crystal families and we list all crystal families of spaces E^1, E^2, E^3, E^4, E^5 (paper XII; Weigel & Veysseyre, 1993), space E^6 (paper XIII) and space E^7 (paper XIV).

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References

Brown, H., Bülow, R., Neubüser, J., Wondratschek, H. & Zassenhaus, H. (1978). *Crystallographic Groups of Four-Dimensional Space.* New York: Wiley-Interscience.

Phan, T., Veysseyre, R. & Weigel, D. (1988). *Acta Cryst.* **A44**, 627–637.

Veysseyre, R., Weigel, D., Phan, T. & Effantin, J. M. (1984). *Acta Cryst.* **A40**, 331–337.

Weigel, D., Phan, T. & Veysseyre, R. (1986). *Acta Cryst.* **A43**, 294–304.

Weigel, D. & Veysseyre, R. (1991). *C. R. Acad. Sci.* Sér. 2, **313**, 481–486.

Weigel, D. & Veysseyre, R. (1993). *Acta Cryst.* **A49**, 486–492.

Weigel, D., Veysseyre, R., Phan, T., Effantin, J. M. & Billiet, Y. (1984). *Acta Cryst.* **A40**, 323–330.

Acta Cryst. (1993). A**49**, 486–492

Crystallography, Geometry and Physics in Higher Dimensions. XII. Counting and General Nomenclature of N-Dimensional Crystal Families: Application from One- to Five-Dimensional Spaces

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Abstract

In paper XI [Veysseyre, Weigel & Phan (1993). *Acta Cryst.* A**49**, 481–486], the definition was given of the geometrically Z-irreducible and the geometrically Z-reducible crystal families of the n-dimensional space and a general method was described for constructing all crystal families. In this paper, systematic rules are stated for giving names to the crystal families and these are listed for spaces E^1, E^2, E^3, E^4 and E^5.

Introduction

The aim of this paper is to explain how the geometrically Z-reducible (gZ-red.) property of a crystal family enables us to give a name to these families through some examples. This paper is divided into three sections:

(i) § 1 is concerned with the counting (*i.e.* the number) of all crystal families from one- to seven-dimensional spaces;

(ii) § 2 expresses strict rules that lead us to assign correct names to the gZ-red. crystal families of E^n and mainly to spaces of dimensions one to five;

(iii) § 3 explains the connection between the transitive crystallographic rotations (Bertaut, 1988) and the geometrically Z-irreducible (gZ-irr.) families and explains the choice of their names too (for the one- to five-dimensional spaces).

I. Counting of all crystal families of space E^n

The study of all partitions of space E^n into subspaces that are two-by-two orthogonal enables us to describe all gZ-red. crystal families. The type of the gZ-reducibility is given by the dimension of each space occurring in the splitting of space E^n and by the type of the irreducibility of the crystal family (Veysseyre, Weigel & Phan, 1993).

For instance, if we consider the partition $E^6=E^3\oplus E^2\oplus E^1$, we say that the type of crystal family built in this way is $(3)+(2)+1$, where (3) means 3 or $\overline{1,1,1}$ and (2) means 2 or $\overline{1,1}$. A general formula will be given § 2. In fact, we can select in space E^3 one of the two gZ-irr. families and in space E^2 one of the three gZ-irr. families (Table 1) and obviously in space E^1 the only gZ-irr. family, *viz* the segment. So, we

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